CONCORDANT SEMIGROUPS AND BALANCED CATEGORIES

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Abstract. Here we define balanced categories it is shown that the category of left [right] ideals of a concordant semigroup are balanced categories.

A non-empty set S with associative binary operation is called a semigroup, if S contains an identity element, then it is called a monoid. A subset of S which is closed under the induced binary operation is called a subsemigroup. An element $e \in S$ is called an idempotent if $e^2 = e$ and the set of all idempotents is S will be denoted by $E(S)$. An element $a \in S$ is called regular if there exists an element $a' \in S$ such that $aa'a = a$, if every element os S is regular then S is a regular semigroup. A subset I of a semigroup S is called left (right, two sided) ideal if $SI \subseteq I\,(IS \subseteq I,\, SIS \subseteq I)$. In the study of the structure of semigroups ideals and idempotents play vital role.

1. Prelimanires

First we recall the basic definitions and results semigroups needed in the sequel. Let S be a semigroup and $a \in S$ the smallest left ideal containing a is $Sa \cup \{a\}$ and is called the principal left ideal generated by a, written as S^1a. Also aS^1 is the principal right ideal generated by a and S^1aS^1 the two sided principal ideal generated by a. Next we define certain equivalence relations called Green's relations on semigroups using these principal ideals.

Definition 1. *Let a, b be elements in a semigroup S, then*

- *$a\mathcal{L}b$ if and only if they generate the same principal left ideal. ie., $S^1a = S^1b$.*
- *$a\mathcal{R}b$ if and only if they generate the same principal right ideal. ie., $aS^1 = bS^1$.*
- *$a\mathcal{J}b$ if and only if they generate the same principal two sided ideal. ie., $S^1aS^1 = S^1bS^1$.*

The intersection of $\mathcal{R}$ and $\mathcal{L}$ is of great importance and is denoted by $\mathcal{H}$ and their join by $\mathcal{D}$. These equivalence relations are termed as Green's equivalences and are significant in the study of semigroups.

Note that in a regular semigroups each $\mathcal{L}$ - class and each $\mathcal{R}$ - class contains idempotents, ie., regular semigroups may be described as those class of semigroups in which each $\mathcal{L}$ - class and each $\mathcal{R}$ - class contains idempotents.

Let S be a semigroup. For $a, b \in S$ are said to be $\mathcal{L}^*$ and $\mathcal{R}^*$ related on S if and only if for all $x, y \in S^1$, $ax = ay \Leftrightarrow bx = by$ $[xa = ya \Leftrightarrow xb = yb]$ and $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, $\mathcal{D}^* = \mathcal{L}^* \cup \mathcal{R}^*$. These are equivalence relations and are called

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generalized Green's relations. Note that these relations coincides with te Green's relations if the semigroup S is regular.

Definition 2. *A semigroup S is called abundant if each $\mathcal{L}^*-$ class and each $\mathcal{R}^*-$ class contains idempotents.*

Remark 1. *Two elements $a, b \in S$ are $\mathcal{L}^*\,[\mathcal{R}^*]$ related on S if and only if there exists a semigroup $\bar{S}$ containing S such that a, b are $\mathcal{L}\,[\mathcal{R}]$ related on $\bar{S}$.*

Definition 3. *A semigroup S is said to be idempotent connected if for each $a \in S$ and for each $a^+ \in \mathcal{R}^* \cap E(S)$ and $a^* \in \mathcal{L}^* \cap E(S)$ there is a bijection $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ satisfying*

$$xa = a(x\alpha) \text{ for all } x \in \langle a^+ \rangle$$

where for $e \in E(S)$, $\langle e \rangle$ denotes the subsemigroup of S generated by the set $\{f \in E(S) : f \leq e\}$.

1.1. **Categories.** We assume familiarity with basic definitions and results in category theory. However we recall the following ; a morphism f in a category $\mathcal{C}$ is a monomorphism (mono, in short) if for $g, h \in \mathcal{C}$, $gf = hf$ implies $g = h$, that is the morphism is right cancellable and a monomorphism $f \in \mathcal{C}(c, c')$ is split if there exists a $g \in \mathcal{C}$ with $fg = 1_c$. Dually a morphism $f \in \mathcal{C}$ is an epimorphism (epi,in short) if f is left cancellable and is split epi if there exixts a $g \in \mathcal{C}(c', c)$ such that $gf = 1_{c'}$.

Definition 4. *A morphism $f \in \mathcal{C}$ is called a balanced morphism if it is both mono and epi.*

Two morphisms $f, g \in \mathcal{C}$ are equivalent if there exists $h, k \in \mathcal{C}$ such that $f = hg$ and $g = kf$.

If there exists a choice of subobjects P (ie., an equivalence class of monomorphisms) in the category $\mathcal{C}$, then the pair $(\mathcal{C}, P)$ is called a category with subobjects (or a choice of subobjects in $\mathcal{C}$). A category $\mathcal{C}$ is said to have factorization property if every $f \in \mathcal{C}$ can be expressed as $f = pm$ where p is an epimorpism and m is an embedding. A factorization of the form $f = qj$ where q is an epimorphism and j is an inclusion is called a canonnical factorization. A category $\mathcal{C}$ is said to have factorization property if and only if every morphism in $\mathcal{C}$ admits a cannonical factorization.

Let $\mathcal{C}$ be a category with subobjects , for each $c \in v\mathcal{C}$, we denote by $\langle c \rangle_{\mathcal{C}}$ the full subcategory of $\mathcal{C}$ whose objects are subobjects of $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is an ideal of $\mathcal{C}$ if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ such that $\langle c \rangle \subseteq \mathcal{C}'$ for all $c \in v\mathcal{C}'$. The ideal $\langle c \rangle_{\mathcal{C}}$ is called the ideal generated by c.

2. Balanced Categories

A factorization of a morphism $f \in \mathcal{C}$ of the form euj where e is a retraction u is a balanced morphism and j is an inclusion is called a balanced factorization of f. In particular if u is split then u is an isomorphism and such a factorization is called a normal factorization.

Proposition 1. *Let $\mathcal{C}$ be a category with factorization property in which every inclusion splits, then*

(1) *if f admits balanced facrorizations euj and $e'u'j'$ then $eu = e'u'$ and $j = j'$.*

(2) *if f is an epi morphism then every balanced factorization of f if exists is of the form eu, where e is a retraction and u is a balanced morphism.*
(3) *if f is a monomorphism then f admts a unique balanced factorization.*

Definition 5. *Let $u \in \mathcal{C}(c,d)$ be a balanced morphism and let $c_1 \subset c$, define $T_u : \langle c \rangle \to \langle d \rangle$ by*

$$T_u(c_1) = im(j^c_{c_1} u) \, for \; each \, c_1 \subseteq c$$
$$T_u(j^c_{c_1}) = j^d_{T_u(c_1)}$$

such that $j^c_{c_1} u = u_{c_1} j^d_{T_u}$ where u_{c_1} is the epimorphic component of $j^c_{c_1} u$. Then the balanced morphism u is called proper if T_u is an isomorphism.

Definition 6. *A category $\mathcal{C}$ is called balanced if it satisfies the following*

(1) *every inclusion in $\mathcal{C}$ splits*

(2) *every morphism in $\mathcal{C}$ admits a balancede factorization*

(3) *every balanced morphism is proper*

(4) *if $f \in \mathcal{C}$ is such that $f = je$ then f admits a normal factorization*

2.1. **Semigroup of Balanced Cones.** Let $\mathcal{C}$ be a balanced category. A balanced cone γ is a mapping from v$\mathcal{C}$ to $\mathcal{C}$ such that

(1) there is a $c_\gamma \in v\mathcal{C}$ such that for each $c \in v\mathcal{C}$, $\gamma_c : c \to c_\gamma$ and if $c_1 \subseteq c$ then $\gamma_{c_1} = j^c_{c_1} \gamma_c$.

(2) there exists atleast one $c \in v\mathcal{C}$ such that $\gamma_c : c \to c_\gamma$ is a balanced morphism.

For each balanced cone γ in $\mathcal{C}$ we define the M-set of γ as

$$M\gamma = \{c \in v\mathcal{C} : \gamma_c \, \text{is a balanced morphism}\,\}.$$

If $f \in \mathcal{C}(c_\gamma, d)$ is an epimorphism then the map

$$\gamma * f : c \to \gamma_c f \, \text{for all} \, c \in \mathcal{C}$$

is a balanced cone. We denote the set of all balanced cones in $\mathcal{C}$ by $\mathcal{TC}$.

Theorem 1. *For any balanced cones $\gamma^1, \gamma^2 \in \mathcal{TC}$ the binary composition defined by*

$$\gamma^1 \cdot \gamma^2 = \gamma^1 * (\gamma^2_{c_{\gamma^1}})^\circ$$

is a closed and associative binary operation on $\mathcal{TC}$ with respect to which $\mathcal{TC}$ is a semigroup.

Theorem 2. *The semigroup $\mathcal{TC}$ is a concordant semigroup.*

For a balanced category $\mathcal{C}$, its balanced dual is the category denoted by $B^*\mathcal{C}$ is the full subcategory of $\mathcal{C}^*$ with

$$vB^*\mathcal{C} = \{H(\epsilon, -) \,:\, \epsilon \in E(\mathcal{TC}$$

and morphisms are natural transformations between such functors. It can be seen that $B^*\mathcal{C}$ is a category with subobjects in which the inclusion is the inclusion amoung se valued functors.

2.2. **Ideal Categories of Concordant Semigroups.** Let S be a concordant semigroup and let $L(S)$ denotes the category whose objects are principal left ideals generated by idempotents and morphisms are translations $\rho : Se \to Se'$, $e, e' \in E(S)$ is a morphism in if and only if $\rho = \rho_t|Se$ where $\rho_t : s \mapsto st$ is a right translation induced by $t \in eSe'$, that is

$$L(S)(Se, Se') = \{\rho(e, t, e') : t \in eSe'\}.$$

It is easy to verify that

- $\rho(e, t, e')$ is a monomorphism if and only if $e\mathcal{R}^*t$
- $\rho(e, t, e')$ is an epimorphism if and only if $t\mathcal{L}^*e'$
- $\rho(e, t, e')$ is a balanced morphism if and only if $e\mathcal{R}^*t\mathcal{L}^*e'$

and for all $g \in E(R^*_u) \cap \omega(e)$ and $h \in E(L^*_u)$, we have

$$\rho(e, u, f) = \rho(e, g, g) \cdot \rho(g, u, h) \cdot \rho(h, h, f)$$

is a balanced factorization of $\rho(e, u, f)$. ie., every morphism in the category $L(S)$ has a balanced factorization and every such factorization arises in this way hence $L(S)$ is a balanced category. Dually we can also define the balanced category $R(S)$. Since $R(S) = L(S^{op})$, it is also a balanced category.

Theorem 3. *Let $\mathcal{C}$ be a balanced category. Then $F : \mathcal{C} \to L(\mathcal{TC})$ is an isomorphism of balanced categories. Consequently, a small category $\mathcal{C}$ with subobjects is balanced if and only if it is isomorphic to the category $L(S)$ of principal left ideals of a concordant semigroup.*

3. Cross connections

Definition 7. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with sub objects. A functor $F : \mathcal{C} \to \mathcal{D}$ is called a local isomorphism if F is inclusion preserving , fully faithfull and for each $c \in v\mathcal{C}$, $F \mid \langle c\rangle$ is an isomorphism of the ideal $\langle c\rangle$ onto $\langle F(c)\rangle$.*

Suppose $\mathcal{C}$ and $\mathcal{D}$ be balanced categories. A local isomorphism $\Gamma : \mathcal{C} \to B^*\mathcal{D}$ is called a connection of $\mathcal{C}$ with $\mathcal{D}$. Given a connection $\Gamma : \mathcal{D} \to B^*\mathcal{C}$ and $\Gamma(d)$ is an h-functor for all $d \in \mathcal{D}$. Since $M\Gamma(d)$ is nonempty for each $d \in v\mathcal{D}$ there is a $c \in v\mathcal{D}$ with $c \in M\Gamma(d)$. Let C_Γ denotes the ideal of $\mathcal{C}$ with

$$vC_\Gamma = \{c \in v\mathcal{C} : c \in M\Gamma(d) \text{ : for some } d \in v\mathcal{D}\}$$

.

Theorem 4. *Let $\Gamma : \mathcal{D} \to B^*\mathcal{C}$ be connection of balanced categories $\mathcal{C}$ with $\mathcal{D}$ and let C_Γ denotes the ideal of $\mathcal{C}$, then there exists a unique connection $\Gamma^* : C_\Gamma \to B^*\mathcal{D}$ such that*

$$c \in M\Gamma(d) \Leftrightarrow d \in M\Gamma^*(c)$$

and a natural isomorphism $\chi_\Gamma : \Gamma(-,-) \to \Gamma^*(-,-)$.

Definition 8. *A cross connection of balanced category $\mathcal{C}$ with $\mathcal{D}$ is a triplet (Γ, Δ, χ) where $\Gamma : \mathcal{D} \to B^*\mathcal{C}$ and $\Delta : \mathcal{C} \to B^*\mathcal{D}$ are local isomorphisms such that*

$$c \in M\Gamma(d) \Leftrightarrow d \in M\Delta(c)$$

and $\chi : \Gamma(-,-) \to \Delta(-,-)$ is a natural isomorphism.

Remark 2. *If the connection* $\Gamma : \mathcal{D} \to B^*\mathcal{C}$ *satisfies the condition*

$$vC_\Gamma = v\mathcal{C}$$

then $(\Gamma, \Gamma^*, \chi_\Gamma)$ *is a cross connection.*

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